JOURNAL OF APPROXIMATION THEORY 2, 29–49 (1969)

On *Lg*-Splines[1]

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1. INTRODUCTION

The theory of polynomial spline functions interpolating prescribed data at knots $x_1 < x_2 < \ldots < x_k$ has proved amenable to extension in several directions. Schoenberg [*15*] initiated the departure from polynomial splines by considering trigonometric interpolating splines. Greville [*8*] was the first to study splines associated with a general linear differential operator interpolating simple data. Ahlberg, Nilson and Walsh later supplemented Greville's treatment in [*1*].

On the other hand, Ahlberg and Nilson [*2*] introduced polynomial splines interpolating arbitrary derivatives at the knots $\{x_i\}_1{}^k$ (Hermite–Birkhoff data), and Schoenberg [*16*] subsequently refined this theory. Later authors, e.g., Karlin and Ziegler [*11*] and Schultz and Varga [*17*], considered splines associated with a general linear differential operator, interpolating consecutive derivatives (Hermite data).

The purpose of this paper is to investigate splines corresponding to a general differential operator which interpolate general data, including, e.g., Hermite–Birkhoff data. To accomplish this, we employ a Hilbert space approach similar to that found in Golomb [*6*] and Anselone and Laurent [*3*]. The advantages of this approach are that existence and uniqueness results are clearly distinguished, characterization of the splines is facilitated, and explicit computational algorithms, involving inversion of positive definite m-banded matrices, are obtained. In addition, we carry out the theory for data prescribed at an infinite number of points, determine the best approximation of linear functionals in the sense of Sard, and investigate splines where interpolation is relaxed to inequality constraints on certain linear functionals. Hilbert space

[1] This work was sponsored by the Mathematics Research Center, United States Army, Madison, Wisconsin, under Contract No.: DA-31-124-ARO-D-462.

techniques have also been employed by Atteia [4], de Boor and Lynch [5], and Sard [13], but not for the cases of the general data considered here.

We now define the notion of an Lg-spline. Let L be a linear differential operator of the form

$$L = \sum_{j=0}^{m} a_j \left(\frac{d}{dx}\right)^j, \quad a_m(x) \neq 0 \text{ on } [a,b], \quad a_j \in C^j[a,b], \quad 0 \leqslant j \leqslant m. \tag{1.1}$$

We denote by $\mathscr{H}_2^m(a,b)$ the Hilbert space of real-valued functions $f \in C^{m-1}[a,b]$, such that $f^{(m-1)}$ is absolutely continuous and $Lf \in \mathscr{L}_2(a,b)$, with the inner product

$$(f,g)_{\mathscr{H}_2^m} = \sum_{j=0}^{m-1} f^{(j)}(a)\, g^{(j)}(a) + \int_a^b Lf\, Lg.$$

Suppose $\Lambda = \{\lambda_i\}_1^n$ is a sequence of continuous linear functionals, linearly independent on $\mathscr{H}_2^m$, and suppose $r = (r_1, r_2, \ldots, r_n) \in E^n$.

DEFINITION 1.1. A function $s \in \mathscr{H}_2^m$ is called an Lg-spline interpolating r with respect to Λ provided it solves the following minimization problem:

$$\|Ls\|_{\mathscr{L}_2} = \min_{f \in U(r)} \|Lf\|_{\mathscr{L}_2},$$

$$U(r) = \{f \in \mathscr{H}_2^m : \lambda_j f = r_j,\ 1 \leqslant j \leqslant n\}. \tag{1.2}$$

We have chosen the terminology "Lg-splines" since it has been customary to call polynomial splines interpolating Hermite–Birkhoff data g-splines, and splines associated with a general differential operator, L-splines.

The variational problem (1.2) can also be considered in $\mathscr{L}_2(\mathfrak{R})$, provided the coefficients of the differential operator (1.1) are sufficiently regular. In this case, a solution of the minimization problem is referred to as a natural Lg-spline. In the following we shall be concerned primarily with the case of an interval $[a,b]$.

2. EXISTENCE AND UNIQUENESS OF Lg-SPLINES

If L is a linear differential operator as in (1.1), then it is easily seen to define a bounded linear operator from $\mathscr{H}_2^m(a,b)$ onto $\mathscr{L}_2(a,b)$. Its null space $N = N_L$ is of dimension m and is spanned by functions $\{u_i\}_1^m$ in $C^m[a,b]$. We formulate the main result of this section in

THEOREM 2.1. *There exists an* $s \in \mathscr{H}_2^m$ *satisfying* (1.2). *A function* $s \in U(r)$ *solves* (1.2) *if and only if*

$$\int_a^b Ls\, Lg = 0 \qquad \text{for every} \qquad g \in U(0). \tag{2.1}$$

Moreover, any two solutions of (1.2) *corresponding to a prescribed* $r \in E^n$ *differ by a function in* N, *and* (1.2) *possesses a unique solution if and only if* $N \cap U(0) = (0)$.

Proof. In view of the linear independence of the $\{\lambda_i\}_1^n$, the closed flat $U(r)$ of (1.2) is a (non-empty) translate of the subspace $U(0) = \{f \in \mathscr{H}_2^m : \lambda_j f = 0, 1 \leqslant j \leqslant n\}$. The facts that $U(0)$ is closed and N is finite dimensional imply that $U(0) + N$ is closed. Since L is onto, it follows from Lemma 2.1 of [*6*] that $LU(0)$ and thus $LU(r)$ is closed, and hence the minimization problem (1.2) possesses a solution. Viewing (1.2) as a projection problem in $\mathscr{L}_2$, the orthogonality relation (2.1) is immediate.

Conversely, if (2.1) holds for some $s \in U(r)$, then it follows easily that s is a solution of (1.2). Indeed,

$$\begin{aligned} \int (Lf)^2 &= \int (Ls)^2 + 2 \int (Ls)(Lf - Ls) + \int (Lf - Ls)^2 \\ &= \int (Ls)^2 + \int (Lf - Ls)^2 \end{aligned}$$

for every $f \in U(r)$, and thus $\int (Ls)^2 \leqslant \int (Lf)^2$ for all $f \in U(r)$. Clearly (2.1) implies that any two solutions of (1.2) differ by an element of N, and hence (1.2) possesses a unique solution if and only if $N \cap U(0) = (0)$.

COROLLARY 2.2. *The class of Lg-splines*

$$\mathscr{S} = \mathscr{S}(L, \Lambda) = \{s : s \text{ satisfies } (1.2) \text{ for some } r = (r_1, \ldots, r_n) \in E^n\}$$

is a finite dimensional linear subspace of $\mathscr{H}_2^m$, *with* $\dim \mathscr{S} = n + \dim N \cap U(0)$. *Moreover,* $N \subset \mathscr{S}$.

Proof. To show linearity, let s and $\tilde{s}$ be splines in $\mathscr{S}$ interpolating r and $\tilde{r}$, and let $s^* = \gamma s + \tilde{\gamma}\tilde{s}$, where $\gamma, \tilde{\gamma}$ are real. We claim: s^* satisfies (1.2) for $r^* = \gamma r + \tilde{\gamma}\tilde{r}$. Indeed, $s^* \in U(r^*)$, and by (2.1), if $g \in U(0)$,

$$\int Lg\, Ls^* = \gamma \int Lg\, Ls + \tilde{\gamma} \int Lg\, L\tilde{s} = 0.$$

But then, by Theorem 2.1, s^* satisfies (1.2) for r^*.

We now show the finite dimensionality of $\mathscr{S}$ by constructing an explicit basis. Let s_j be a solution of (1.2) corresponding to $r =$ the jth row of the $n \times n$ identity matrix, i.e.

$$\lambda_i s_j = \delta_{ij}, \qquad 1 \leqslant i \leqslant n, \qquad 1 \leqslant j \leqslant n. \tag{2.2}$$

LEMMA 2.3. *Suppose* $\dim(N \cap U(0)) = d$, *and let* $\{v_i\}_1^d$ *be a basis for* $N \cap U(0)$. *Then* $\dim \mathscr{S} = n + d$, *and the functions* $\{s_i\}_1^n \cup \{v_i\}_1^d$ *form a basis for* $\mathscr{S}$.

Proof. Let $s \in \mathscr{S}$, and set $\Delta = s - \sum_{i=1}^{n} (\lambda_i s) s_i$. Clearly $\Delta \in U(0) \cap \mathscr{S}$, and thus, by (2.1),

$$0 = \int L\Delta L\Delta = \int (L\Delta)^2.$$

But this implies that $\Delta \in N \cap U(0)$, and hence

$$s = \sum_{i=1}^{n} (\lambda_i s) s_i + \sum_{i=1}^{d} \gamma_i v_i$$

for some $\{\gamma_i\}_1{}^d$. We now establish the linear independence of $\{s_i\}_1{}^n \cup \{v_i\}_1{}^d$. Suppose

$$\theta = \sum_{i=1}^{n} \beta_i s_i + \sum_{i=1}^{d} \gamma_i v_i = 0.$$

Then $0 = \lambda_j \theta = \beta_j$, $1 \leqslant j \leqslant n$, and since the $\{v_i\}_1{}^d$ are linearly independent, we also have $\gamma_j = 0$, $1 \leqslant j \leqslant d$. The fact that $N \subset \mathscr{S}$ follows trivially from the characterizing orthogonality relation (2.1). This completes the proof of Corollary 2.2.

Following [*16*] we call the interpolation problem (1.2) poised with respect to L provided that $N \cap U(0) = (0)$. In this case there exists a unique Lg-spline interpolating r with respect to Λ, and Lemma 2.3 asserts that $\mathscr{S}$ is of dimension n and is spanned by the $\{s_j\}_1{}^n$ themselves.

3. Characterization of Lg-Splines Interpolating EHB Data

The class of linear functionals

$$\mathscr{L}^{(m)} = \left\{\lambda : \lambda f = \sum_{i=0}^{m-1} \int_a^b f^{(i)}(x)\, d\mu_i(x), \qquad \mu_i \text{ of bounded variation}\right\}$$

provides an example of the type of linear functionals which are suitable for Λ. For this section it is convenient to single out two special choices of Λ. We say that Λ generates a Hermite–Birkhoff (HB) interpolation problem, if to each $\lambda_i \in \Lambda$ there corresponds a pair (x_i, j_i) such that $\lambda_i f = f^{(j_i)}(x_i)$, where $a \leqslant x_i \leqslant b$ and $0 \leqslant j_i \leqslant m-1$. On the other hand, if for each $\lambda_i \in \Lambda$, $\lambda_i f = \sum_{j=0}^{m-1} \alpha_{ij} f^{(j)}(x_i)$, where α_{ij} are real numbers, we say that Λ generates an Extended–Hermite–Birkhoff problem (EHB) provided that the vectors $\alpha_i = (\alpha_{i,0}, \alpha_{i,1}, \ldots, \alpha_{i,m-1})$ defining linear functionals λ_i associated with the same point are linearly independent. It is easy to see that an (HB) interpolation problem is a special case of an (EHB) interpolation problem. We also remark that any n linear functionals defining an (EHB) interpolation problem are linearly independent over $\mathscr{H}_2{}^m$ by virtue of their form and the assumption on the α_i's.

The main result of this section is Theorem 3.6 which characterizes Lg-splines interpolating EHB data. The key tool for this purpose is the orthogonality relation (2.1). We begin by defining a knot. An Lg-spline s interpolating $r = (r_1, r_2, \ldots, r_n)$ with respect to $\Lambda = \{\lambda_i\}_1^n$ is said to have a knot at the point $x \in [a,b]$, provided some $\lambda_i \in \Lambda$ involves evaluation of some jth derivative, $0 \leqslant j \leqslant m-1$, at x. We now show that any solution s of (1.2) corresponding to EHB data Λ satisfies $L^*Ls(x) = 0$ in the intervals between its knots $a \leqq x_1 < x_2 < \ldots < x_k \leqslant b$, where L^* is the formal adjoint of L defined by

$$L^* f = \sum_{j=0}^{m} (-1)^j (a_j f)^{(j)}.$$

Let $g \in C_c^\infty(x_i, x_{i+1})$. Then $g \in U(0)$ automatically, and hence (2.1) yields

$$0 = \int_a^b Ls\, Lg = \int_{xi}^{xi+1} sL^* Lg,$$

upon integration by parts. It follows by well-known arguments that $L^*Ls(x) = 0$ on (x_i, x_{i+1}).

Next we prove that $Ls(x) = 0$ for $a < x < x_1$ and $x_k < x < b$. First we notice that the above discussion applies equally well to show that $L^*Ls(x) = 0$, and thus that Ls is continuous for $0 < x < x_1$ and $x_k < x < b$. Suppose now that $Ls(\xi) \neq 0$ for some $a < \xi < x_1$. Let

$$\tilde{s}(x) = \begin{cases} s(x), & x_1 < x \leqslant b, \\ u(x), & a \leqslant x \leqslant x_1, \end{cases}$$

where $u(x) \in N$ satisfies $u^{(j)}(x_1) = s^j(x_1)$, $0 \leqslant j \leqslant m-1$. Then $\tilde{s} \in \mathscr{H}_2{}^m$ and $\int (L\tilde{s})^2 < \int (Ls)^2$. But since $\lambda_i s = \lambda_i \tilde{s}$ for all $\lambda_i \in \Lambda$, this contradicts the fact that s minimizes (1.2). A similar proof holds for the interval (x_k, b).

We now suppose that $x \in (a,b)$ and that s is an Lg-spline interpolating EHB data. If $\epsilon > 0$ is sufficiently small and $g \in U(0) \cap C_c^\infty(x-\epsilon, x+\epsilon)$, then integrating (2.1) by parts gives

$$0 = \int_{x-\epsilon}^{x+\epsilon} Ls\, Lg = \sum_{i=0}^{m-1} g^{(i)}(x)[O_i s]_x, \tag{3.1}$$

where

$$O_i s = \sum_{j=0}^{m-i-1} (-1)^{j+1} (a_{j+i+1} Ls)^{(j)}$$

and the notation $[\cdot]_x$ is defined by $[\phi]_x = \phi(x+) - \phi(x-)$. A relation analogous to (3.1) also holds for the points a and b, with $[\phi]_a = \phi(a+)$ and $[\phi]_b = -\phi(b-)$.

To facilitate the characterization of splines interpolating EHB data, it is convenient to rearrange (3.1). Let $\alpha = (\alpha_{ij})_{0,0}^{l-1,m-1}$, $l \leqslant m$, be of rank l, and let $\tilde{\alpha}$

be an $m \times m$ nonsingular matrix obtained by augmenting α. We denote by $\eta = (\eta_{ij})$ the inverse of the adjoint of $\tilde{\alpha}$. The proof of the following lemma is trivial.

LEMMA 3.1. *Let*

$$M_i = \sum_{j=0}^{m-1} \tilde{\alpha}_{ij} \left(\frac{d}{dx}\right)^i \quad \textit{and} \quad R_i = \sum_{j=0}^{m-1} \eta_{ij} O_j, \quad 0 \leqslant i \leqslant m-1.$$

Then

$$\sum_{i=0}^{m-1} g^{(i)}(x)[O_i s]_x = \sum_{i=0}^{m-1} M_i g(x)[R_i s]_x \tag{3.2}$$

for all $g \in \mathscr{H}_2{}^m$ *and* $s \in \mathscr{S}$.

THEOREM 3.2. *Suppose s is an Lg-spline interpolating $\{r_i\}_1{}^n$ with respect to $\Lambda = \{\lambda_i\}_1{}^n$, where Λ generates an* EHB *interpolation problem. In particular, suppose $x \in [a,b]$ is a knot of s and that there are $l(x)$ linear functionals in Λ which involve some derivatives of s at x, of the form*

$$M_i^{(x)} s(x) = \sum_{j=0}^{m-1} \alpha_{ij}(x) s^{(j)}(x), \quad 0 \leqslant i \leqslant l(x) - 1, \tag{3.3}$$

where $\{\alpha_i = (\alpha_{i,0}, \ldots, \alpha_{i,m-1})\}_0^{l(x)-1}$ are assumed to be linearly independent. Then

$$[R_j^{(x)} s]_x = 0, \quad l(x) \leqslant j \leqslant m-1, \tag{3.4}$$

where the $R_j^{(x)}$ are defined as in Lemma 3.1.

Proof. Assume $x \in (a,b)$ and fix $j, l(x) \leqslant j \leqslant m-1$. Choose $\epsilon > 0$ such that x is the only knot of s in the interval $(x-\epsilon, x+\epsilon)$. It is easily seen that there exists a function $g \in C_c^\infty(x-\epsilon, x+\epsilon)$ satisfying $\bar{g} = \tilde{\alpha}^{-1} I_j$, where I_j is the jth column of the $m \times m$ unit matrix and $\bar{g} = (g(x), \ldots, g^{(m-1)}(x))^T$.

By construction $M_i^{(x)} g(x) = \delta_{ij}$, $0 \leqslant i \leqslant m-1$, and $g \in U(0)$. Now combining (3.1) and (3.2) yields $0 = [R_j^{(x)} s]_x$. The cases when $l(a) > 0$ or $l(b) > 0$ are handled similarly.

We note that since $s \in C^{m-1}$,

$$([O_{m-1} s]_x, \ldots, [O_0 s]_x)^T = \xi([s^{(m)}]_x, \ldots, [s^{(2m-1)}]_x)^T,$$

where ξ is lower triangular with $\pm a_m{}^2(x)$ on the main diagonal. Since

$$([R_0^{(x)} s]_x, \ldots, [R_{m-1}^{(x)} s]_x)^T = \eta([O_0 s]_x, \ldots, [O_{m-1} s]_x)^T,$$

it follows that the equations (3.4) represent $m - l(x)$ linearly independent relations among the $\{[s^{(j)}]_x\}_m^{2m-1}$ at the point x. The choice of the augmentation $\tilde{\alpha}$ of α in Lemma 3.1 is not critical inasmuch as any two choices lead to equivalent sets of $m - l(x)$ linearly independent relations.

The following corollary is an easy consequence of either Theorem 3.2 or its proof.

COROLLARY 3.3. *Suppose s is an Lg-spline corresponding to an* HB *interpolation problem. If the lth derivative* $(0 \leqslant l \leqslant m-1)$ *evaluated at the knot x is not involved in the* HB *data, then* $[O_l s]_x = 0$.

COROLLARY 3.4. *Suppose s is an Lg-spline corresponding to an* HB *interpolation problem, and suppose ν denotes the order of the highest derivative specified at a knot $x \in (a,b)$. Then $[s^{(j)}]_x = 0$ for $0 \leqslant j \leqslant 2m-2-\nu$.*

Proof. Since $s \in C^{m-1}$, it follows trivially that $[s^{(j)}]_x = 0$ for $0 \leqslant j \leqslant m-1$. Now suppose the conclusion is valid for all j satisfying $0 \leqslant j \leqslant p < 2m-2-\nu$. In view of the assumption that the highest derivative specified at x is of order ν, we deduce that the $2m-p-2^{nd}$ is not involved. Hence by Corollary 3.3

$$0 = [O_{2m-p-2}\, s]_x = \sum_{j=0}^{p-m+1} (-1)^{j+1} [(a_{j+2m-p-1} Ls)^{(j)}]_x.$$

Using Leibnitz's rule and the inductive assumption $[s^{(j)}]_x = 0$ for $0 \leqslant j \leqslant p$, this reduces to $0 = (-1)^{p-m}[a_m{}^2 s^{(p+1)}]_x$, and the assertion follows since $a_m{}^2(x) \neq 0$.

In the special case that $L = (d/dx)^m$, the Lg-spline interpolating HB data is called a g-spline (see [*16*]). We have

COROLLARY 3.5. *Let s be a g-spline interpolating* HB *data, and suppose the lth derivative* $(0 \leqslant l \leqslant m-1)$ *is not specified at a knot x. Then* $[s^{(2m-l-1}]_x = 0$.

Proof. By Corollary 3.3,

$$0 = [O_l s]_x = \sum_{j=0}^{m-l-1} (-1)^{j+1} [(a_{j+l+1} Ls)^{(j)}]_x.$$

But for $L = (d/dx)^m$ we have $a_j \equiv 0$, $0 \leqslant j \leqslant m-1$, and hence this reduces to $0 = (-1)^{m-l}\, [s^{(2m-l-1)}]_x$.

We close this section with the following characterization theorem.

THEOREM 3.6. *Let s be an Lg-spline interpolating* EHB *data $\{r_i\}_1{}^n$ with respect to $\{\lambda_i\}_1{}^n$. Then*

$L^*Ls(x) = 0$	*if x is not a knot, and $x \in (a,b)$.*		(3.5a)
$\lambda_i s \quad = r_i$	$1 \leqslant i \leqslant n$.		(3.5b)
$[R_i^{(x)} s]_x = 0$	$l(x) \leqslant i \leqslant m-1$, *if x is a knot.*		(3.5c)
$Ls(x) \quad = 0$	*for $a < x < x_1$ and $x_k < x < b$.*		(3.5d)

Conversely, any function $s \in \mathscr{H}_2{}^m$ *satisfying* (3.5) *is an Lg-spline interpolating* $\{r_i\}_1{}^n$ *with respect to* $\{\lambda_i\}_1{}^n$. *In particular, if the* EHB *interpolation problem is poised then* (3.5) *uniquely characterizes the spline.*

Proof. The direct implications have already been established above. By Theorem 2.1, the converse follows immediately from the easily verifiable relation

$$\int_a^b Ls(Ls - Lf) = 0, \qquad \text{for any } f \in U(r).$$

In the case of natural Lg-splines interpolating EHB data, Theorem 3.6 holds with $a = -\infty$ and $b = +\infty$ in (3.5d).

4. A Basis for $L\mathscr{S}$

In this section we derive a basis for the space $L\mathscr{S}$ which leads to a method for the computation of Lg-splines involving the inversion of a positive definite m-banded matrix. This is of more than academic interest inasmuch as the usual bases for splines frequently lead to ill-conditioned systems of equations with full matrices (cf. [*3*], [*9*]). The analysis is similar to that in [*3*], and in the development we obtain a number of lemmas of independent interest.

As remarked in §2, the L of (1.1) is a bounded linear operator from $\mathscr{H}_2{}^m(a,b)$ onto $\mathscr{L}_2(a,b)$. Let L^a be the adjoint operator from $\mathscr{L}_2(a,b)$ into $\mathscr{H}_2{}^m(a,b)$, defined by

$$(Ly, x)_{\mathscr{L}_2} = (y, L^a x)_{\mathscr{H}_2{}^m}. \tag{4.1}$$

By the closed range theorem (see [*18*], page 205),

$$R(L^a) = \overline{R(L^a)} = N_L{}^\perp \qquad \text{and} \qquad N_{L^a} = (R(L))^\perp = (0). \tag{4.2}$$

Thus $(L^a)^{-1}$ also exists.

Throughout the remainder of this section we assume that $\Lambda = \{\lambda_i\}_1{}^n$ consists of linear functionals on $\mathscr{H}_2{}^m$ in the class $\mathscr{L}^{(m)}$ defined in §3, and that $N \cap U(0) = (0)$.

Lemma 4.1. $N \cap U(0) = (0)$ *if and only if there exists a subsequence* $\{\tilde{\lambda}_i\}_1{}^m$ of Λ *which is linearly independent over N.*

Proof. Suppose $\{\tilde{\lambda}_i\}_1{}^m$ is linearly independent over N, and let $u \in N \cap U(0)$. Then $u = \sum_1^m \gamma_j u_j$ for some $\{\gamma_j\}_1{}^m$, and

$$0 = \tilde{\lambda}_i u = \sum_{j=1}^m \gamma_j \tilde{\lambda}_i u_j, \qquad 1 \leqslant i \leqslant m.$$

By the assumption of the linear independence, the matrix$(\tilde{\lambda}_i u_j)_{1,1}^{m,m}$ is non-singular, and hence $\gamma_j = 0$, $1 \leqslant j \leqslant m$, i.e., $u \equiv 0$.

Conversely, suppose no m λ's in Λ are linearly independent over N, i.e., the matrix $A=(\lambda_i u_j)_{1,1}^{n,m}$ has rank less than m. Then the mapping $A:E^m \to E^n$ possesses a non-trivial vector $\gamma=(\gamma_1,\ldots,\gamma_m)$ in its null space. But then $u=\sum_{i=1}^m \gamma_i u_i \in N \cap U(0)$, and $u \not\equiv 0$.

In the remainder of this section we shall use a slightly different inner product for $\mathscr{H}_2{}^m$ which is equivalent to the one defined in §1. Namely, we set

$$(f,g)_{\mathscr{H}_2{}^m}=\sum_{i=1}^m \tilde{\lambda}_i f \tilde{\lambda}_i g+\int_a^b Lf\, Lg\, dx, \tag{4.3}$$

where $\{\tilde{\lambda}_i\}_1{}^m$ is linearly independent over N, as in Lemma 4.1. By the Riesz representation theorem, there exist functions $k_j \in \mathscr{H}_2{}^m$ such that

$$\lambda_j f=(f,k_j)_{\mathscr{H}_2{}^m},\ 1\leqslant j\leqslant n.$$

LEMMA 4.2. *$\{k_j\}_1{}^n$ forms a basis for $\mathscr{S}$.*

Proof. To show that $k_j \in \mathscr{S}$, we show that it satisfies (1.2) with $r_i=\lambda_i k_j$, $1\leqslant i\leqslant n$. If $\phi \in U(r)$ with $r=(r_1,\ldots,r_n)$, then

$$0=\lambda_j(\phi-k_j)=(\phi-k_j,k_j)_{\mathscr{H}_2{}^m}.$$

But by the definition of the inner product, this yields

$$0=\sum_{i=1}^m \tilde{\lambda}_i(\phi-k_j)\tilde{\lambda}_i k_j+\int_a^b (L\phi-Lk_j)Lk_j\,dx=\int_a^b (L\phi-Lk_j)Lk_j\,dx,$$

and thus by Theorem 2.1, $k_j \in \mathscr{S}$.

As remarked at the end of §2, the assumption of poisedness assures that $\dim \mathscr{S}=n$, and thus the proof will be completed by establishing that $\{k_j\}_1{}^n$ is linearly independent. Suppose $\sum_{j=1}^n \gamma_j k_j \equiv 0$. Then if $s_1,s_2,\ldots,s_n$ are the fundamental splines of (2.2),

$$0=\sum_{j=1}^n \gamma_j(k_j,s_i)_{\mathscr{H}_2{}^m}=\gamma_i, \qquad 1\leqslant i\leqslant n.$$

LEMMA 4.3. $\mathscr{H}_2{}^m=\mathscr{S}\oplus U(0)$.

Proof. Since $\mathscr{S}$ is finite dimensional, it suffices to show that $U(0)=\mathscr{S}^\perp$. By (2.1), clearly $U(0)\subset\mathscr{S}^\perp$. On the other hand, if $f\in\mathscr{S}^\perp$, then by Lemma 4.2,

$$\lambda_i f=(f,k_i)_{\mathscr{H}_2{}^m}=0, \qquad 1\leqslant i\leqslant n,$$

i.e. $f\in U(0)$.

A simple corollary of the above lemma is the fact that for any $f\in\mathscr{H}_2{}^m$, the function $s\in\mathscr{S}$ obtained by projecting f onto $\mathscr{S}$ satisfies (1.2) with $r_i=\lambda_i f$, $1\leqslant i\leqslant n$.

LEMMA 4.4. $(L\mathscr{S}^{\perp})^{\perp} = L\mathscr{S}$.

Proof. By Lemma 4.3, $\mathscr{S}^{\perp} = U(0)$ and so $L\mathscr{S} \subset (L\mathscr{S}^{\perp})^{\perp}$ follows trivially from (2.1). Conversely, if $f \in (LU(0))^{\perp}$, then $0 = \int fLg$ for every $g \in U(0)$. Let $\tilde{f} \in \mathscr{H}_2{}^m$ be such that $L\tilde{f} = f$. Then

$$0 = \int L\tilde{f} Lg = (g, \tilde{f})_{\mathscr{H}_2{}^m},$$

since $g \in U(0)$. This asserts that $\tilde{f} \in U(0)^{\perp} = \mathscr{S}$, and thus $f \in L\mathscr{S}$.

LEMMA 4.5. $L^a L\mathscr{S} = N^{\perp} \cap \mathscr{S}$.

Proof. Suppose $y \in L^a L\mathscr{S}$, i.e., $y = L^a x$, $x \in L\mathscr{S} = (L\mathscr{S}^{\perp})^{\perp}$. By the definition (4.1) of L^a, we have $x \in (L\mathscr{S}^{\perp})^{\perp}$ if and only if $L^a x \in \mathscr{S}$. Therefore $y \in \mathscr{S}$, and since $y \in R(L^a) = N^{\perp}$, we have shown that $L^a L\mathscr{S} \subset N^{\perp} \cap \mathscr{S}$. For the converse, suppose $y \in N^{\perp} \cap \mathscr{S}$, i.e., $y \in R(L^a) \cap \mathscr{S}$. Then $y = L^a x \in \mathscr{S}$ and as noted above $x \in L\mathscr{S}$, i.e., $y \in L^a L\mathscr{S}$.

It is now convenient to introduce a function $\hat{\theta}(\xi, x)$ such that every $f \in \mathscr{H}_2{}^m$ has a representation

$$f(\xi) = \sum_{i=1}^{m} b_i u_i(\xi) + \int_a^b \hat{\theta}(\xi, x) Lf(x)\, dx. \tag{4.4}$$

Following [8], let $W(x) = (u_j^{(i-1)}(x))_{1,1}^{m;m}$ be the Wronskian matrix of the functions $\{u_i\}_1{}^m$ which span N. The assumptions on L assure that $|W(x)| \neq 0$ for $x \in [a,b]$. Define $\theta(\xi, x) = \bar{u}(\xi)[W(x)]^{-1} I_m$, where $\bar{u}(\xi) = (u_1(\xi), \ldots, u_m(\xi))$, and I_m is the last column of the unit matrix of order m. Then the function

$$\hat{\theta}(\xi, x) = \begin{cases} \theta(\xi, x) & \text{for } \xi > x, \\ 0 & \text{for } \xi \leqslant x, \end{cases}$$

leads to (4.4) (cf. [8]). For example, in the special case $L = (d/dx)^m$, we have $\hat{\theta}(\xi, x) = (\xi - x)_+^{m-1}$, and (4.4) is just Taylor's formula with integral remainder. For later reference we remark that although $\hat{\theta}(\cdot, x) \notin \mathscr{H}_2{}^m$, nevertheless $\lambda_i \hat{\theta}(\cdot, x)$ is well defined for almost all $x \in [a,b]$ if $\lambda_i \in \mathscr{L}^{(m)}$.

Before exhibiting a basis for $L\mathscr{S}$, we establish the existence of vectors $\beta_j = (\beta_{j1}, \ldots, \beta_{jn})$, $1 \leqslant j \leqslant n - m$, satisfying

$$\{\beta_j\}_1^{n-m} \text{ is linearly independent,} \tag{4.5a}$$

$$\sum_{i=1}^{n} \beta_{ji} \lambda_i u_l = 0, \qquad 1 \leqslant l \leqslant m, \qquad 1 \leqslant j \leqslant n - m. \tag{4.5b}$$

To accomplish this, consider the mapping $A^T : E^n \to E^m$ defined by the matrix $A = (\lambda_i u_l)_{1,1}^{n;m}$. By the poisedness assumption, A is of rank m, and hence there exist linearly independent vectors $\beta_j \in E^n$, $j = 1, 2, \ldots, n-m$, which span its null space, i.e., satisfy (4.5).

THEOREM 4.6. *The functions*

$$f_j(x) = \sum_{i=1}^{n} \beta_{ji} \lambda_i \hat{\theta}(\cdot, x) \qquad 1 \leqslant j \leqslant n-m \tag{4.6}$$

form a basis for $L\mathscr{S}$.

Proof. First we notice that

$$\dim(L\mathscr{S}) = \dim((L^a)^{-1}(N^\perp \cap \mathscr{S})) = \dim(N^\perp \cap \mathscr{S}),$$

by Lemma 4.5 and the fact that $(L^a)^{-1}$ exists, Moreover,

$$\dim(N^\perp \cap \mathscr{S}) = \dim(\mathscr{S}) + \dim(N \cap \mathscr{S}^\perp) - \dim(N).$$

But $N \cap \mathscr{S}^\perp = (0)$, inasmuch as $N \subset \mathscr{S}$, and since $\dim(\mathscr{S}) = n$, it follows that $\dim(L\mathscr{S}) = n - m$.

To show that $f_j \in L\mathscr{S}$, it suffices to establish $L^a f_j \in L^a L\mathscr{S} = N^\perp \cap \mathscr{S}$. By (4.1) and (4.5b),

$$(L^a f_j, \phi)_{\mathscr{H}_2{}^m} = (f_j, L\phi)_{\mathscr{L}_2} = \sum_{i=1}^{n} \beta_{ji} \lambda_i (\hat{\theta}, L\phi)_{\mathscr{L}_2} = \sum_{i=1}^{n} \beta_{ji} \lambda_i \phi = 0$$

for $\phi \in U(0)$. (Here we have relied on a standard application of Fubini's theorem, yielding $(\lambda_i \hat{\theta}, L\phi)_{\mathscr{L}_2} = \lambda_i(\hat{\theta}, L\phi)_{\mathscr{L}_2}$). Hence $L^a f_j \in U(0)^\perp = \mathscr{S}$. On the other hand,

$$(L^a f_j, \psi)_{\mathscr{H}_2{}^m} = (f_j, L\psi)_{\mathscr{L}_2} = 0$$

if $\psi \in N$, i.e., $L^a f_j \in N^\perp$ also.

Finally, we verify the linear independence of $\{f_j\}_1^{n-m}$. Suppose $\sum_{j=1}^{n-m} \gamma_j f_j \equiv 0$. Then if s_l is the spline satisfying (2.2),

$$0 = \sum_{j=1}^{n-m} \gamma_j (f_j, Ls_l)_{\mathscr{L}_2} = \sum_{j=1}^{n-m} \gamma_j \sum_{i=1}^{n} \beta_{ji} \lambda_i (\hat{\theta}, Ls_l)_{\mathscr{L}_2}$$

$$= \sum_{j=1}^{n-m} \gamma_j \sum_{i=1}^{n} \beta_{ji} [\lambda_i s_l + \lambda_i \phi_l] = \sum_{j=1}^{n-m} \gamma_j \beta_{jl}, \qquad 1 \leqslant l \leqslant n. \qquad (\phi_l \in N).$$

In view of (4.5a), the matrix $(\beta_{jl})_{1,1}^{n-m,n}$ is of rank $n-m$ and hence this implies $\gamma_j = 0$, $1 \leqslant j \leqslant n-m$.

By virtue of the above theorem, any Lg-spline interpolating $\{r_i\}_1{}^n$ with respect to $\{\lambda_i\}_1{}^n = \Lambda \subset \mathscr{L}^{(m)}$, satisfies $Ls(x) = \sum_{i=1}^{n-m} c_i f_i(x)$, where $\{c_i\}_1^{n-m}$ is uniquely determined by the system of equations

$$\sum_{i=1}^{n-m} c_i(f_i,f_j)_{\mathscr{L}_2} = (Ls,f_j)_{\mathscr{L}_2} = (\beta_j, r)_{l_2}, \qquad 1 \leqslant j \leqslant n-m, \tag{4.7}$$

since the Grammian $\det(f_i, f_j) \neq 0$. The inner product (f_i, f_j) is given by

$$(f_i,f_j)_{\mathscr{L}_2} = \sum_{\nu=1}^{n} \sum_{\mu=1}^{n} \beta_{i\nu}\beta_{j\mu}(\lambda_\nu \hat{\theta}(\cdot\,,x), \lambda_\mu \hat{\theta}(\cdot\,,x))_{\mathscr{L}_2}. \tag{4.8}$$

Finally, by (4.4) and the above representation of Ls,

$$s(x) = \sum_{i=1}^{n-m} c_i(\hat{\theta}(x,y), f_i(y))_{\mathscr{L}_2} + \sum_{i=1}^{m} q_i u_i(x),$$

where $\{q_i\}_1{}^m$ is uniquely determined from the system

$$\sum_{i=1}^{m} q_i \tilde{\lambda}_j u_i = \tilde{r}_j - \sum_{i=1}^{n-m} c_i \tilde{\lambda}_j(\hat{\theta}, f)_{\mathscr{L}_2}, \qquad 1 \leqslant j \leqslant m, \tag{4.9}$$

where $\{\tilde{\lambda}_j\}_1{}^m$ is as in Lemma 4.1, and $\{\tilde{r}_j\}_1{}^m$ is the corresponding sequence of prescribed values.

It is important to notice that the basis $\{f_j\}_1^{n-m}$ for $L\mathscr{S}$ defined in (4.6) depends on the choice of $\{\beta_j\}_1^{n-m}$. In general, the matrix of the system (4.7) will not be m-banded. However, we shall now show that the β_j's can be chosen to satisfy (4.5) and to yield a sparse matrix in (4.7). First we notice that

$$\begin{aligned}(f_i,f_j)_{\mathscr{L}_2} &= \sum_{\nu=1}^{n} \sum_{\mu=1}^{n} \beta_{i\nu}\beta_{j\mu}(\rho_\nu, \rho_\mu)_{\mathscr{L}_2},\\ &= \beta_i \rho \beta_j{}^T,\end{aligned}$$

where $\rho = (\rho_{ij}) = ((\rho_i, \rho_j)_{\mathscr{L}_2})$ and $\rho_i(x) = \lambda_i \hat{\theta}(\cdot\,, x)$. Since ρ is the Grammian of $\{\rho_i\}_1{}^n$, it is nonnegative definite. By (4.6), $L\mathscr{S}$ is contained in the span of $\{\rho_i\}_1{}^n$, and hence the rank r of ρ is at least $n-m$. It follows that the null space N_ρ of ρ has dimension $n - r \leqslant m$. We conclude that ρ defines a pseudo-inner product

$$(x, y)_\rho = x\rho y^T, \qquad x = (x_1, \ldots, x_n), \qquad y = (y_1, \ldots, y_n),$$

for which $(x, x)_\rho = 0$ if and only if $x \in N_\rho$, i.e., the set of vectors in E^n with zero ρ-length is of dimension at most $n - r \leqq m$.

THEOREM 4.7. *There exists a* $\{\beta_i{}^*\}_1^{n-m}$ *satisfying* (4.5), *with*

$$(\beta_j{}^*, \beta_i{}^*)_\rho = (\beta_i{}^*, \beta_j{}^*)_\rho = 0, \qquad i \neq j, \qquad 1 \leqslant i \leqslant n-2m, \qquad 1 \leqslant j \leqslant n-m. \tag{4.10}$$

Then $((f_i, f_j)_{\mathscr{L}_2})$ *consists of the identity matrix of order* $n-2m$ *in the upper left-hand corner, an* $m \times m$ *matrix in the lower right-hand corner and otherwise possesses zero elements.*

Proof. We observe that the result is immediate if $n-m \leqslant m$. Suppose now that $n-m > m$. Let $\{\beta_i\}_1^{n-m}$ satisfy (4.5). We now proceed by induction to obtain $\{\beta_i^*\}_1^{n-m}$ with the property (4.10). Since $\{\beta_i\}_1^{n-m}$ is linearly independent there exists a β_j, say β_1, of nonzero ρ-length. Then we may take $\beta_1^* = \beta_1/\sqrt{(\beta_1, \beta_1)_\rho}$.

Suppose now that $\beta_1^*, \ldots, \beta_i^*$, $i < r-m$, have been obtained by orthonormalizing $\beta_1, \ldots, \beta_i$. We now select β_{i+1}^*. Consider

$$\left\{\beta_j - \sum_{\nu=1}^{i} \beta_\nu^*(\beta_\nu^*, \beta_j)_\rho\right\}_{j=i+1}^{n-m},$$

which is easily seen to be linearly independent. Now since $n-m-i > n-r$, at least one of these vectors, say

$$\hat{\beta}_{i+1} = \beta_{i+1} - \sum_{\nu=1}^{i} \beta_\nu^*(\beta_\nu^*, \beta_j)_\rho,$$

has nonzero ρ-length. Then we take $\beta_{i+1}^* = \hat{\beta}_{i+1}/\sqrt{(\hat{\beta}_{i+1}, \hat{\beta}_{i+1})_\rho}$.

Assume now that $\{\beta_j^*\}_1^{r-m}$ has been obtained by orthonormalizing $\{\beta_j\}_1^{r-m}$. Then we choose

$$\beta_j^* = \beta_j - \sum_{\nu=1}^{r-m} (\beta_j, \beta_\nu^*)\beta_\nu^*, \qquad r-m+1 \leqslant j \leqslant n-m.$$

It is easily verified that this inductive choice of $\{\beta_i^*\}_1^{n-m}$ satisfies the theorem.

5. Interpolation on Infinite Sets

In this section we shall consider a generalized interpolation problem on arbitrary closed sets B of $\mathfrak{R}$ as in [7]. Indeed, let $\{M_i^{(x)}\}_{i=0}^{l(x)-1}$ be a collection of operators defined for each $x \in B$, with $1 \leqslant l(x) \leqslant m$, where m is the order of the differential operator of (1.1) and

$$M_i^{(x)} = \sum_{k=0}^{m-1} \alpha_{ik}(x) \left(\frac{d}{dx}\right)^k, \qquad i = 0, 1, \ldots, l(x)-1,$$

with $M_0^{(x)}\phi(x) = \phi(x)$. It is assumed that, for each x, $\{M_i^{(x)}\}_{i=0}^{l(x)-1}$ is a linearly independent sequence of rank $l(x)$, i.e., the $l(x)$ vectors $\alpha_i(x) = (\alpha_{ik}(x))$ are linearly independent. For each $x \in B$, the sequence $\{M_i^{(x)}\}_{i=0}^{l(x)-1}$ can be augmented to form a sequence $\{M_i^{(x)}\}_{i=0}^{m-1}$ of rank m as in Lemma 3.1. Let $\{R_i^{(x)}\}_{i=0}^{m-1}$ be the sequence of operators defined there. Then for each $x \in B - B'$, where B' denotes the set of limit points of B,

$$\int_J LuL\phi = \int_J L^*Lu\cdot\phi + \sum_{i=0}^{m-1} M_i^{(x)}\phi(x)\,[R_i^{(x)}u]_x, \tag{5.1}$$

where J is any interval such that $J \cap B = \{x\}$, ϕ is an arbitrary function in $C_c^\infty(J)$, and $u \in C^{2m}(J - \{x\})$.

We seek a solution to the generalized interpolation problem:

$$L^* LF(x) = 0 \qquad x \in \mathfrak{R} - B \tag{5.2a}$$

$$M_i^{(x)} F(x) = M_i^{(x)} f(x) \qquad i = 0, 1, \ldots, l(x) - 1, \qquad x \in B, \tag{5.2b}$$

$$F \in \mathscr{H}_L{}^m(\mathfrak{R}) \cap \mathscr{H}_{\text{loc}}^{2m}(\mathfrak{R} - B) \cap \mathscr{D}_B, \tag{5.2c}$$

where f is any prescribed function in $\mathscr{H}_L{}^m(\mathfrak{R})$, and

$\mathscr{H}_{\text{loc}}^m(\mathfrak{R}) = \{u : u^{(m-1)}$ is locally absolutely continuous with $u^{(m)}$ locally square integrable$\}$,

$\mathscr{H}_L{}^m(\mathfrak{R}) = \{u \in \mathscr{H}_{\text{loc}}^m(\mathfrak{R}) : Lu \in \mathscr{L}_2(\mathfrak{R})\}$,

$$\mathscr{D}_B = \{u \in \mathscr{H}_L{}^m(\mathfrak{R}) \cap \mathscr{H}_{\text{loc}}^{2m}(\mathfrak{R} - B) : [R_i^{(x)} u]_x = 0, \qquad x \in B - B', \quad \text{and} \quad i = l(x), \ldots, m-1\}.$$

There exists a solution to the interpolation problem (5.2) which is unique under additional hypotheses. This result slightly generalizes results in [7] and the proof will not be duplicated. The approach is to consider a minimization problem in $\mathscr{H}_L{}^m(\mathfrak{R})$, as in §2, over a closed flat defined by (5.2b), and to identify the solution of the minimization problem with the solution of (5.2) through the use of (5.1). Any two solutions of (5.2) differ by a null solution for L in the special case that the problem:

$$\begin{aligned} L^* F(x) &= 0 \qquad x \in \mathfrak{R} - B' \\ F &\in \mathscr{L}_2(\mathfrak{R}), \end{aligned} \tag{5.3}$$

has only the identically zero solution. In this case we shall say that L satisfies property (5.3). Then the solution of (5.2) is unique if any null solution for L, which satisfies (5.2b) homogeneously, must be identically zero, which is true if N_L is spanned by a Tchebycheff system, and B contains at least m points.

It is well to point out that for any $x \in B'$ the values $f^{(k)}(x)$, $k = 1, 2, \ldots, m-1$, are determined by the values of f on B in a neighborhood of x, if $f \in \mathscr{H}_L{}^m(\mathfrak{R})$. Thus, the operators $M_i^{(x)}$, $0 < i \leqslant l(x) - 1$, are significant only for $x \in B - B'$. We summarize the preceding discussion in

THEOREM 5.1. *There exists a real-valued function satisfying* (5.2). *If L has property* (5.3), *then any two functions satisfying* (5.2) *differ by a null solution for L. Finally, if N_L is spanned by a Tchebycheff system, and L has property* (5.3), *then* (5.2) *has a unique solution.*

If L has property (5.3) then any solution of (5.2) minimizes the expression

$$\min_{F \in \mathscr{U}} \|LF\|_{\mathscr{L}_2}, \tag{5.4}$$

where $\mathscr{U}$ is the flat defined by

$$\mathscr{U} = \{F \in \mathscr{H}_L{}^m(\Re) : M_i^{(x)} F(x) = M_i^{(x)} f(x), \forall\, x \in B\}.$$

In this case, consistent with our earlier terminology, we may call any solution of (5.4) an Lg-spline with a characterization described by (5.2). In particular, the operator $L = (d/dx)^m$ has property (5.3) and, in addition, the monomials $1, x, \ldots, x^{m-1}$ form a Tchebycheff system so that in this case the unique Lg-spline satisfying (5.4) is uniquely characterized by (5.2). The special case when B is compact merits consideration; say, $-\infty < a = \min_{x \in B} x$ and $b = \max_{x \in B} x < \infty$. In this case, if the condition

$$LF(x) = 0 \qquad \text{for} \qquad x < a \qquad \text{and} \qquad x > b \tag{5.2d}$$

is satisfied, then (5.2) is equivalent to (5.4) without the additional hypothesis that L has property (5.3). We then say that F is a natural Lg-spline, which is clearly consistent with our earlier terminology if B is a finite point set.

Finally, if L has property (5.3) and N_L is spanned by a Tchebycheff system then the Lg-spline characterized by (5.2) can be approximated by splines with finitely many knots. Indeed, let

$$X = \{x_1, x_2, \ldots\}$$

be a sequence dense in B and consider the truncations

$$X_n = \{x_1, \ldots, x_n\}.$$

From previous results, the interpolation problem:

$$\begin{aligned} L^* L s_n(x) &= 0 \qquad x \in \Re - X_n, \\ M_i s_n(x) &= M_i f(x) \qquad 0 \leqslant i \leqslant l(x) - 1, \qquad x \in X_n, \\ s_n &\in \mathscr{H}_L{}^m(\Re) \cap \mathscr{H}^{2m}_{\mathrm{loc}}(\Re - X_n) \cap \mathscr{D}_{X_n}, \end{aligned} \tag{5.5}$$

has a unique solution for $n \geqslant m$. Then, as in [7], we now have

THEOREM 5.2. *If L has property* (5.3) *and N_L is spanned by a Tchebycheff system, then $s_n \to F$ in $\mathscr{H}_L{}^m(\Re)$, where F uniquely solves* (5.2a, b, c). *In particular $s_n^{(k)}$ converges uniformly to $F^{(k)}$ on compact subsets of $\Re$ for* $0 \leqslant k \leqslant m - 1$.

In the special case that B is compact, the sequence s_n of (5.5) satisfying $Ls_n(x) = 0$ if $x < \min_{x \in X_n} x$ or $x > \max_{x \in X_n} x$, converges to the unique solution of (5.2a, b, c, d) as in the above theorem, without the assumption that L has property (5.3).

6. APPROXIMATION OF LINEAR FUNCTIONALS

Let $\mathscr{L}^{(m)}$ be the class of linear functionals defined in §3. We now consider the problem of approximating a linear functional $\lambda_0 \in \mathscr{L}^{(m)}$ by a linear combination of $\{\lambda_i\}_1^n \subset \mathscr{L}^{(m)}$ in the sense of Sard (cf. [*13*], [*14*]). For $\alpha = (\alpha_1, \ldots, \alpha_n)$, set

$$R(\alpha) = \lambda_0 - \sum_{i=1}^{n} \alpha_i \lambda_i.$$

Then for all $f \in \mathscr{H}_2^m$, (4.4) yields

$$R(\alpha) f = R\psi + \int_a^b K(\alpha, x) Lf(x)\, dx, \tag{6.1}$$

where $\psi \in N_L$ and $K(\alpha, x) = R(\alpha)\hat{\theta}(\cdot, x)$.

THEOREM 6.1. *Let $\{\lambda_i\}_1^n \subset \mathscr{L}^{(m)}$ be such that $N \cap \mathrm{U}(0) = (0)$ and let $s_1, s_2, \ldots, s_n$ be the Lg-splines satisfying $\lambda_i s_j = \delta_{ij}$. Then among all α such that $R(\alpha)$ annihilates N, the minimum of*

$$\int_a^b [K(\alpha, x)]^2\, dx \tag{6.2}$$

is uniquely attained for α^* given by

$$\alpha_i^* = \lambda_0 s_i, \qquad 1 \leqslant i \leqslant n,$$

i.e., $\sum_{i=1}^n \alpha_i^* \lambda_i$ is the best approximation to λ_0 in the sense of Sard. Moreover, for each $f \in \mathscr{H}_2^m$,

$$\left[\sum_{i=1}^{n} \alpha_i^* \lambda_i\right] f = \lambda_0 s,$$

where s is the Lg-spline interpolating

$$r_i = \lambda_i f, \qquad 1 \leqslant i \leqslant n. \tag{6.3}$$

Proof. First we notice that $R(\alpha^*) s = 0$ for every $s \in \mathscr{S}$. Indeed, $R(\alpha^*)$ annihilates the basis $\{s_j\}_1^n$ of $\mathscr{S}$, as is easily seen from the relation

$$R(\alpha^*) s_j = \lambda_0 s_j - \sum_{i=1}^{n} (\lambda_i s_j) \lambda_0 s_i = \lambda_0 s_j - \lambda_0 s_j = 0.$$

Next we show that $\Delta(\cdot) = K(\alpha^*, \cdot) - K(\alpha, \cdot) \in L\mathscr{S}$, whenever $R(\alpha)$ annihilates N. By Theorem 2.1, it suffices to show that

$$\int_a^b \Delta(x) Lg(x)\, dx = 0 \qquad \text{for every } g \in \mathrm{U}(0).$$

But

$$\int_a^b \Delta(x) Lg(x)\, dx = \sum_{i=1}^{n} (\alpha_i - \alpha_i^*) \lambda_i \int_a^b \hat{\theta}(\cdot, x) Lg(x)\, dx$$

$$= \sum_{i=1}^{n} (\alpha_i - \alpha_i^*) \lambda_i (g + p) = 0,$$

since $p \in N$ and $g \in \mathrm{U}(0)$.

Now let $s \in \mathscr{S}$ be such that $Ls = \Delta$. Then by (6.1),

$$0 = R(\alpha^*)s = \int_a^b K(\alpha^*, x) Ls(x)\,dx$$
$$= \int_a^b K(\alpha^*, x)[K(\alpha^*, x) - K(\alpha, x)]\,dx,$$

which implies that

$$\int_a^b (K(\alpha, x))^2\,dx = \int_a^b (K(\alpha^*, x))^2\,dx + \int_a^b [K(\alpha, x) - K(\alpha^*, x)]^2\,dx, \quad (6.4)$$

i.e., $K(\alpha^*, \cdot)$ minimizes (6.2). We now show that α^* is unique. If α minimizes (6.2), then by (6.4), $K(\alpha^*, x) - K(\alpha, x) = 0$ a.e., and we obtain for each $1 \leqslant j \leqslant n$,

$$0 = \int_a^b [K(\alpha^*, x) - K(\alpha, x)] Ls_j(x)\,dx = \sum_{i=1}^n (\alpha_i - \alpha_i^*)\lambda_i \int_a^b \hat{\theta}(\cdot, x) Ls_j(x)\,dx$$

$$= \sum_{i=1}^n (\alpha_i - \alpha_i^*)\lambda_i(s_j + p) = \alpha_j - \alpha_j^*, \quad \text{i.e., } \alpha = \alpha^*.$$

This completes the proof, since the last statement of the theorem is evident.

Alternately, one may consider the problem (cf. [*5*])

$$\min_\alpha \left\| \lambda_0 - \sum_{i=1}^n \alpha_i \lambda_i \right\|. \quad (6.5)$$

Clearly the minimum is uniquely attained by the linear functional $\bar{\lambda}$ corresponding to the representer Pk_0 obtained by projecting the representer k_0 of λ_0 onto $\mathscr{S}$, i.e., $\bar{\lambda}f = (f, Pk_0)_{\mathscr{H}_2{}^m}$. But by the simple properties of projection operators we have

$$\bar{\lambda}f = (f, Pk_0)_{\mathscr{H}_2{}^m} = (Pf, k_0)_{\mathscr{H}_2{}^m} = \lambda_0 s,$$

where s is the spline interpolating f as in (6.3). It follows that λ is the best approximation of λ_0 in the sense of Sard.

7. Inequality Constraints

The preceding theory is capable of even further generalization. In particular, we shall consider the following minimization problem, where L, $\mathscr{H}_2{}^m$ and Λ are as in §1, and $\underline{r} = (\underline{r}_1, \ldots, \underline{r}_n)$, $\bar{r} = (\bar{r}_1, \ldots, \bar{r}_n) \in E^n$ with $\underline{r} \leqslant \bar{r}$:

$$\|Ls\|_{\mathscr{L}_2} = \min_{f \in \mathscr{U}(\underline{r}, \bar{r})} \|Lf\|_{\mathscr{L}_2},$$
$$\mathscr{U}(\underline{r}, \bar{r}) = \{f \in \mathscr{H}_2{}^m : \underline{r}_j \leqslant \lambda_j f \leqslant \bar{r}_j, \quad 1 \leqslant j \leqslant n\}. \quad (7.1)$$

We shall show below that (7.1) admits of a solution which may be called an Lg-spline interpolating $(\underline{r}, \bar{r})$ with respect to Λ. We also derive a characterization theorem for s, similar to Theorem 3.6. Recently, Ritter [*12*] employed quadratic programming methods to investigate the minimization problem (7.1) in the special case when $L=(d/dx)^m$ and Λ corresponds to Hermite–Birkhoff type linear functionals. By the use of the variational approach, this section readily provides extensions of his work in several directions.

We now state

THEOREM 7.1. *There exists an* $s \in \mathscr{H}_2{}^m$ *satisfying* (7.1). *A function* $s \in \mathscr{U}(\underline{r}, \bar{r})$ *solves* (7.1), *if and only if*

$$\int_a^b Ls\,Lg \leqslant 0 \qquad \text{for all} \qquad g \in \mathscr{U}(s,\underline{r},\bar{r}), \tag{7.2}$$

where $\mathscr{U}(s, \underline{r}, \bar{r}) = \{s - f : f \in \mathscr{U}(\underline{r}, \bar{r}\}$. *Moreover, any two solutions differ by a null function in* N, *and* s *is the unique solution of* (7.1) *if and only if*

$$N \cap \mathscr{U}(s, \underline{r}, \bar{r}) = (0). \tag{7.3}$$

In particular, $N \cap \mathscr{U}(\underline{r} - \bar{r}, \bar{r} - \underline{r}) = (0)$ *implies that* (7.1) *has a unique solution.*

Proof. For the existence of s it suffices to show that $L\mathscr{U}(\underline{r}, \bar{r})$ is a closed, convex subset of $\mathscr{L}_2$. The convexity is obvious. Since the intersection of closed sets is closed, we need only show that the images under L of the sets

$$V_i = \{f \in \mathscr{H}_2{}^m : \lambda_i f \leqslant \bar{r}_i\}, \qquad W_i = \{f \in \mathscr{H}_2{}^m : \lambda_i f \geqslant \underline{r}_i\}$$

are closed for $1 \leqslant i \leqslant n$. To this end we show that the complement $(LV_i)^c$ of LV_i in $\mathscr{L}_2$ is open. The closedness of LW_i follows in the same way. Since L maps $\mathscr{H}_2{}^m$ onto $\mathscr{L}_2$, an arbitrary element of $(LV_i)^c$ may be written in the form Lg, where $g \in V_i{}^c$. We have

$$Lg \in (LV_i)^c \subset (LY_i)^c \qquad \text{where} \qquad Y_i = \{f \in \mathscr{H}_2{}^m : \lambda_i f = \bar{r}_i\}.$$

Since $(LY_i)^c$ is open (see §2), there exists a ball B_i about Lg, entirely contained within $(LY_i)^c$. We claim $B_i \cap LV_i = \varnothing$. Indeed, suppose $Lh \in B_i \cap LV_i$, where $h \in V_i$. Since $\lambda_i g > \bar{r}_i$ and $\lambda_i h \leqslant \bar{r}_i$, it follows that $c = (\bar{r}_i - \lambda_i h)/(\lambda_i g - \lambda_i h)$ satisfies $0 \leqslant c < 1$, and thus, by the convexity of B_i and the definition of c,

$$cLg + (1 - c)Lh \in B_i \cap LY_i$$

which is a contradiction. It follows that $(LV_i)^c$ is open.

The necessity of (7.2) is the usual convexity inequality satisfied by projections in Hilbert space. Conversely, if the quasi-orthogonality relation (7.2) holds, then for any $f \in \mathscr{U}(\underline{r}, \bar{r})$,

$$\int_a^b (Lf)^2 - \int_a^b (Ls)^2 = \int_a^b (Lf - Ls)^2 + 2\int_a^b Ls(Lf - Ls) \geqslant 0,$$

i.e., s solves (7.1).

We now show that the difference of any two solutions s and $\tilde{s}$ of (7.1) is in N. Indeed,

$$0 \leqslant \int_a^b (Ls - L\tilde{s})^2 = \int_a^b Ls(Ls - L\tilde{s}) + \int_a^b L\tilde{s}(L\tilde{s} - Ls) \leqslant 0.$$

Condition (7.3) is clearly necessary for the uniqueness of s. Conversely, suppose s and $\tilde{s}$ are two solutions of (7.1) and suppose (7.3) is satisfied. Then

$$s - \tilde{s} \in N \cap \mathscr{U}(s, \underline{r}, \bar{r}),$$

and hence $s = \tilde{s}$. The final assertion of the theorem is obvious.

We remark that any solution s of (7.1) is a solution of (1.2) with

$$\mathrm{U}(r) = \{f \in \mathscr{H}_2{}^m : \lambda_i f = \lambda_i s,\ 1 \leqslant i \leqslant n\},$$

and thus, in particular, is an Lg-spline. Moreover, if $\underline{r} = \bar{r}$ then (7.1) reduces to (1.2), and hence the class of all solutions of (7.1) as $\underline{r} \leqslant \bar{r}$ ranges over E^n coincides with the class $\mathscr{S}$ of solutions of (1.2) as r varies in E^n.

We now derive a characterization theorem for Lg-splines interpolating $(r, \bar{r})$ with respect to Λ, where Λ consists of EHB-type linear functionals (see §3). In particular, let $a \leqslant x_1 < \ldots < x_k \leqslant b$ be prescribed knots, and let $M_i^{(x_j)}$, $0 \leqslant i \leqslant l(x_j) - 1$, be the linear functionals in Λ of the form (3.3) involving derivatives evaluated at x_j. As in §3 we augment these to obtain sequences $\{M_i^{(x_j)}\}_0^{m-1}$. Let $R_i^{(x_j)}$ be defined from the $M_i^{(x_j)}$ as in Lemma 3.1.

THEOREM 7.2. *Let $s \in \mathscr{U}(\underline{r}, \bar{r})$. Then s satisfies* (7.1) *if and only if* (3.5) *holds and*

$$[R_i^{(x_j)} s]_{x_j} \leqslant 0 \qquad \text{if } M_i^{(x_j)} s(x_j) > \underline{r}_i, \tag{7.4a}$$

$$[R_i^{(x_j)} s]_{x_j} \geqslant 0 \qquad \text{if } M_i^{(x_j)} s(x_j) < \bar{r}_i, \tag{7.4b}$$

$$[R_i^{(x_j)} s]_{x_j} = 0 \qquad \text{if } \underline{r}_i < M_i^{(x_j)} s(x_j) < \bar{r}_i, \tag{7.4c}$$

for $0 \leqslant i \leqslant l(x_j) - 1$, $1 \leqslant j \leqslant k$.

Proof. Since any solution s of (7.1) must be an Lg-spline interpolating $\{\lambda_i s\}_1{}^n$ with respect to Λ, it is clear that conditions (3.5) must be satisfied. Assume now that $M_i^{(x_j)} s(x_j) < \bar{r}_i$ for some $0 \leqslant i \leqslant l(x_j) - 1$. As in the proof of Theorem 3.2 we construct $g \in C_c{}^\infty(x_j - \epsilon, x_j + \epsilon)$ such that

$$[g(x_j), \ldots, g^{(m-1)}(x_j)]^T =$$
$$\tilde{\alpha}^{-1}(x_j)[M_0^{(x_j)} s(x_j), \ldots M_{i-1}^{(x_j)} s(x_j), \bar{r}_i, M_{i+1}^{(x_j)} s(x_j), \ldots, M_{m-1}^{(x_j)} s(x_j)]^T,$$

where $\tilde{\alpha}(x_j)$ is the matrix defining the $\{M_i^{(x_j)}\}_0^{m-1}$ as in Lemma 3.1. Now if $1 < j < k$ we construct $h \in C^\infty$ so that

$$h(t) = \begin{cases} 1 & t \leqslant x_{j-1} + (x_j - x_{j-1})/4, \\ 0 & x_j - (x_j - x_{j-1})/4 \leqslant t \leqslant x_j + (x_{j+1} - x_j)/4, \\ 1 & t \geqslant x_{j+1} - (x_{j+1} - x_j)/4, \end{cases}$$

with a similar definition for $j = 1$ and $j = k$. Then by construction

$$f(t) = g(t) + h(t)s(t) \in \mathscr{U}(\underline{r}, \bar{r}),$$

and

$$M_\nu^{(x_\mu)} f(x_\mu) = M_\nu^{(x_\mu)} s(x_\mu) \qquad \text{for all } 0 \leqslant \nu \leqslant m-1,\ 1 \leqslant \mu \leqslant k, \text{ except when } \nu = i \text{ and } \mu = j,$$

$$M_i^{(x_j)} f(x_j) = \bar{r}_i.$$

Now, integrating by parts as in §3 yields

$$0 \geqslant \int_a^b Ls(Ls - Lf) = \sum_{\mu=1}^{k} \sum_{\nu=0}^{m-1} M_\nu^{(x_\mu)}(s-f)(x_\mu)\,[R_\nu^{(x_\mu)} s]_{x_\mu}$$
$$= M_i^{(x_j)}(s-f)(x_j)\,[R_i^{(x_j)} s]_{x_j} = \{M_i^{(x_j)} s(x_j) - \bar{r}_i\}[R_i^{(x_j)} s]_{x_j},$$

and (7.4b) follows. In a similar manner $M_i^{(x_j)} s(x_j) > \underline{r}_i$ implies $[R_i^{(x_j)} s]_{x_j} \leqslant 0$, while finally, condition (7.4c) is an immediate consequence of (7.4a, b).

For the converse, suppose $f \in \mathscr{U}(\underline{r}, \bar{r})$. Then

$$\int_a^b Ls(Ls - Lf) = \sum_{\mu=1}^{k} \sum_{\nu=0}^{m-1} M_\nu^{(x_\mu)}(s-f)(x_\mu)[R_\nu^{(x_\mu)} s]_{x_\mu}$$

$$= \sum_{\mu=1}^{k} \sum_{\nu=0}^{l(x_\mu)-1} M_\nu^{(x_\mu)}(s-f)(x_\mu)[R_\nu^{(x_\mu)} s]_{x_\mu},$$

since $[R_i^{(x_\mu)} s]_{x_\mu} = 0$ for $l(x_\mu) \leqslant i \leqslant m-1$, by (3.5). Moreover, if $M_i^{(x_\mu)}(s-f)(x_\mu) > 0$ then $M_i^{(x_\mu)} s(x_\mu) > M_i^{(x_\mu)} f(x_\mu) \geqslant \underline{r}_i$ and by (7.4a) $[R_i^{(x_\mu)} s]_{x_\mu} \leqslant 0$, $0 \leqslant i \leqslant l(x_\mu) - 1$. Similarly, $M_i^{(x_\mu)}(s-f)(x_\mu) < 0$ assures $[R_i^{(x_\mu)} s]_{x_\mu} \geqslant 0$ and therefore

$$\int_a^b Ls(Ls - Lf) \leqslant 0,$$

so that by Theorem 7.1, s is an Lg-spline interpolating $\mathscr{U}(\underline{r}, \bar{r})$ with respect to Λ.

Various corollaries analogous to those in §3 may be obtained by specializing either Λ or the operator L. We cite only the following result for g-splines (cf. [*12*]).

COROLLARY 7.3. *Let $L = (d/dx)^m$ and suppose Λ consists of* HB-*type linear functionals. Then $s \in \mathscr{U}(\underline{r}, \bar{r})$ is a solution of* (7.1) *if and only if it satisfies*

$$s^{(2m)}(x) = 0 \qquad \text{if } x \text{ is not a knot,} \tag{7.5a}$$

$$s^{(m)}(x) = 0 \qquad \text{for } x < x_1,\ x > x_k, \tag{7.5b}$$

$$[s^{(2m-1-i)}]_x = 0 \qquad \text{if the } i\text{th derivative is not specified at the knot } x \text{ or if } \underline{r}_i < s^{(i)}(x) < \bar{r}_i, \tag{7.5c}$$

$$(-1)^{m-i}[s^{(2m-1-i)}]_x \geqslant 0 \qquad \text{if } s^{(i)}(x) = \underline{r}_i \text{ at the knot } x, \tag{7.5d}$$

$$(-1)^{m-i}[s^{(2m-1-i)}]_x \leqslant 0 \qquad \text{if } s^{(i)}(x) = \bar{r}_i \text{ at the knot } x. \tag{7.5e}$$

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